Comprehensive exam in Topology (525)

January, 2012.

1. (25 points) Let $X = S^1 \vee S^1$ be the one-point union of two circles. Label the circles "a" and "b".
   (a) Classify all 2-fold covers $p\colon Y \to X$ up to isomorphism of cover. (Do not require Y to be connected.)
   (b) Let $K = X \cup_\phi e^2$ be the space obtained by gluing a 2-disk to X along its boundary along the map $\phi\colon S^1 \to X$ which is the loop $a \cdot b \cdot a^{-1}$ (the composite of the loop a followed by the loop b followed by the reverse of a). Which covers $p\colon Y \to X$ in part (a) are the restriction of some two fold cover $L \to K$ to X?

2. (25 points) Let $X \subset \mathbb{R}^3$ be the space which is the union of the unit sphere $S^2 = \{\, x \mid |x| = 1 \,\}$ and the line segment $L = \{\, (0,0,t) \mid |t| \le 1 \,\}$ connecting north and south poles. Let $x_0 = (0,0,0)$. Describe the group $\pi_1(X, x_0)$ using generators and relations.

3. (25 points) Let A be a non-empty space. The *cone* on A is the quotient space $CA = A \times I / A \times 0$ (i.e., all points in the subspace $A \times 0$ are identified with each other). Write $* \in CA$ for the point which is the image of $A \times 0$. We regard $A \subset CA$ by the inclusion $i\colon A \to CA$ defined by $i(a) = [(a,1)]$.
   (a) Show that CA is contractible.
   (b) Use Eilenberg-Steenrod axioms for singular homology to show that $H_q(CA, A) \approx \tilde{H}_{q-1}A$ for all q.

4. (25 points) Let $X = \{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \,\} \cup \{\, (x,0) \in \mathbb{R}^2 \mid x^2 \le 1 \,\}$ (so X is the union of a cicle with a diameter line-segment). Let Y be the quotient of $X \times [0,1]$ obtained by identifying $X \times 0$ and $X \times 1$ with a 1/2-twist, so that $(x, y, 0)$ is identified with $(-x, -y, 1)$.
   (a) Describe a CW-structure for Y.
   (b) Compute the Euler characteristic of Y.
   (c) Compute the singular homology groups of Y.