Comprehensive exam in Topology (525)

January 19, 2011.

1. Let $p\colon Y \to X$ and $q\colon Z \to X$, be two covering spaces of the same space X, and assume that the spaces X, Y, Z are path connected. Let M be the set of continuous maps $f\colon Y \to Z$ such that $q \circ f = p$. Let $y_0 \in Y$ be a chosen basepoint for Y, and let $x_0 = p(y_0) \in X$.
   (a) Show that the function $\phi\colon M \to q^{-1}(x_0)$ defined by $f \mapsto f(y_0)$ is injective.
   (b) Let $X = S^1$ be the circle. Give an example of covering spaces $p\colon Y \to X$ and $q\colon Z \to X$, with Y and Z path connected, and a base point $y_0 \in Y$, such that $\phi\colon M \to q^{-1}(x_0)$ is not surjective.

2. Let X be a space. The *suspension* of A is the quotient space SA of $X \times I$, obtained by identifying all points in $X \times \{0\}$ to a single point (which I'll call $*_0$), and all points in $X \times \{1\}$ to a single point (which I'll call $*_1$); we require that $*_0 \neq *_1$.

   Using *only* Eilenberg-Steenrod axioms (dimension, sum, exactness, homotopy, excision), show that there are isomorphisms $\widetilde{H}_{n+1}(SX) \approx \widetilde{H}_n(X)$ for all X.

3. Let $n \geq 1$, and let $f\colon S^n \to S^n$ be a continuous self-map of the unit n-sphere. If f has no fixed points, what is the degree of f, and why?

4. Let $X = \{\,(x, y) \in \mathbb{R}^2 \mid x^2 = y^2 \leq 1\,\}$ (so X looks like a letter "X"), and let $\partial X = \{\,(x, y) \in X \mid x^2 = y^2 = 1\,\}$ (the four points at the ends of the arms of the "X"). Let Y be the quotient of $X \times I$ obtained by identifying $X \times 0$ and $X \times 1$ by a 1/2-twist; that is, $(x, 0)$ is identified with $(-x, 1)$ for all $x \in X$. Let $\partial Y \subset Y$ be the subspace which is the image of $\partial X \times I$.
   (a) Describe a CW-structure for Y.
   (b) Compute the singular homology groups of Y and ∂Y.