Comprehensive Exam in Topology
University of Illinois, January 2009

1. (25 points) Let $A = [0,1] \times [0,1]$, and let

$$B = ([0,1] \times \{0\}) \cup \{(0,1)\} \cup \{(1,1)\}.$$

Let $X = A/B$, the space obtained by collapsing the subpsace B to a single point $*$. Calculate $\pi_1(X, *)$.

2. (25 points) For each of the following either give an example or indicate why no example exists.

   (a) A connected space X such that $\pi_1 X$ is a non-trivial finite group.

   (b) A space X such that $H_2 X$ is a non-trivial finite group.

   (c) A retraction from the 2-sphere to the circle on the equator.

   (d) A self-map of the 2-sphere with no fixed points.

3. (25 points) Compute the homology groups $H_n(S^3, A)$ for all $n \geq 0$, where A is a finite set of points in S^3.

4. (25 points) Let X be a CW-complex. The only thing you know about it is that it consists of a single cell in each of the following dimensions: 0, 1, 2, 3, 5.

   (a) Given the above information, what are the possible values for the graded homology group $H_* X$. (It is not enough to consider each homology group $H_q X$ seperately; the possible values of $H_q X$ for a given $q \in \mathbb{Z}$ may constrain the possibilities in other dimensions.) Explain your reasoning.

   (b) Suppose we additionally assume that X is its own universal cover. What can you say now about $H_* X$? Why?