525

# Comprehensive Exam in Topology

University of Illinois, January 2008

**1.** For each of the following, either give explicit examples or indicate why none exists. Justify your answer.

**a)** $X$ and $Y$ path connected spaces and $f : X \to Y$ a a continuous map such that $\pi_1(f)$ is an isomorphism but $H_2(f;\mathbb{Z})$ is not.

**b)** $X$ and $Y$ path connected spaces and $f : X \to Y$ a a continuous map such that $H_1(f;\mathbb{Z})$ is an isomorphism but $\pi_1(f)$ is not.

**c)** $X$ and $Y$ path connected spaces and $f : X \to Y$ a a continuous map such that $\pi_1(f)$ is an isomorphism but $H_1(f;\mathbb{Z})$ is not.

**d)** A continuous map from $S^2$ to itself that has no fixed points.

**e)** A continuous map of $D^3$ to itself that has no fixed points.

**2.**

Let $X$ be a path-connected, locally path-connected and semilocally simply-connected space with a universal cover $\tilde{X} \xrightarrow{p} X$. Prove that if $\tilde{X}$ is compact than $\pi_1(X)$ is finite.

**3.**

Compute $H_*(X;\mathbf{Z})$ if $X$ is the (reduced) double-suspension of $\mathbb{R}P^5$. That is, given a point $r_0 \in \mathbb{R}P^5$ and $s_0 \in S^2$, we take

$$X = S^2 \wedge \mathbb{R}P^5 = \frac{\mathbb{R}P^5 \times S^2}{(r, s_0) \sim (r_0, s_0) \sim (s_0, r)}.$$

**4.** Prove that if $Y$ is a CW complex and $H_4(Y;\mathbf{Z}) \neq 0$ then $Y$ must have a cell in dimension 4. Prove that if $X$ a CW-complex and $H_4(X) = \mathbb{Z}/6\mathbb{Z}$ then $X$ must have a cell in dimension 5 as well as a cell in dimension 4.