525

COMPREHENSIVE EXAM IN TOPOLOGY

JANUARY 2007

On this exam, D^n is the standard n-disk

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\},$$

and $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is its boundary. For example, S^0 is the discrete space $\{\pm 1\}$.

1. Recall that $H_0 X \cong \mathbb{Z}\pi_0 X$. That is, the zeroth homology of X is the free abelian group generated by the set of path components of X. So an element of $H_0 X$ can be written in the form

$$\sum a_i C_i,$$

where $a_i \in \mathbb{Z}$ and $C_i \in \pi_0 X$. Define the *reduced* homology $\bar{H}_0 X$ to be the kernel of the homomorphism

$$\epsilon : H_0 X \to \mathbb{Z}$$

given by

$$\epsilon\left(\sum a_i C_i\right) = \sum a_i.$$

Calculate $\bar{H}_0 S^0$, and calculate the effect on $\bar{H}_0 S^0$ of the map

$$r : S^0 \to S^0$$

given by the formula $r(x) = -x$.

2. In this problem n is an integer greater than 1. Suppose that $(Y, *)$ is a pointed space, that

$$f : S^{n-1} \to Y$$

is a pointed map, and that X is obtained from Y by attaching an n-cell along f. That is, X is the quotient

$$X = \frac{D^n \coprod Y}{S^{n-1} \ni a \sim f(a) \in Y}.$$

Equivalently, X is the pushout in the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ D^n \arrow[r] & X. \end{tikzcd}$$

For each $n \geq 2$, describe the relationship between $\pi_1(Y; *)$ and $\pi_1(X; *)$.

3. Let $p = (1, 0) \in S^1$. Let T be the torus $S^1 \times S^1$, and let W be the subspace $(S^1 \times \{p\}) \cup (\{p\} \times S^1) \subset T$. Show that $T/W \cong S^2$. Is the projection map

$$\pi : T \to S^2$$

homotopic to a constant map? Justify your answer.

4. In the following picture, is there a retraction $r : X \to A$? Is there a retraction $r : X \to B$? Justify your answers.

X is a surface of genus 2.