Comprehensive exam in Topology (525)

August, 2011.

1. (25 points) Let X be the space obtained from an annulus $\{ p \in \mathbb{R}^2 \mid 1 \leq |p| \leq 2 \}$ by identifying each point (x, y) on the inner circle of radius 1 with the point $(-2x, -2y)$ on the outer circle of radius 2.

   (a) Describe a CW-structure for X.

   (b) Compute $\pi_1(X)$.

   (c) Compute $H_*(X)$.

2. (25 points) Let $X = S^1 \times S^1$ (a torus). Classify all the 3-sheeted covering spaces over X, up to equivalence of covering spaces; give justification. (For the purposes of this question, a covering space of X need not be a connected space.)

3. (25 points) Let X be a path connected and locally path connected space X, and let $Y = S^1 \times \cdots \times S^1$, a product of n copies of the circle, with $n \geq 1$.

   Show that if $\pi_1(X, x_0)$ is finite, then every map $f \colon X \to Y$ is null-homotopic.

4. (25 points) Let (X, A) be a pair, $A \neq \varnothing$, and write $j \colon A \to X$ for the inclusion map. Let $Y = X \cup_A CA$ the space obtained as a quotient of $X \amalg CA$ by identifying $i(a)$ with $j(a)$ for each $a \in A$.

   (a) Show that X is a retract of Y if and only if j is homotopic to a constant map.

   (b) Show that $H_*(X, A) \approx H_*(Y, CA)$, where the map is induced by the inclusion $X \subset Y$.

   (c) Show that $H_*(Y, CA) \approx \widetilde{H}_*(Y)$, and thus $H_*(X, A) \approx \widetilde{H}_*Y$. (Here, $\widetilde{H}$ denotes reduced homology.)