Comprehensive Exam in Topology
University of Illinois, August 2010

1. (25 points) Let X be the space obtained by gluing together a Möbius band B and a torus $T = S^1 \times S^1$, by identifying the boundary circle $C \subset B$ of the Möbius band with the circle $C' = S^1 \times \{x_0\}$ inside the torus. Let $G = \pi_1(X)$.

   (a) Describe G in terms of generators and relations.

   (b) Is the group G abelian?

   (c) Compute $H_1(X)$.

2. (25 points) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere; let $E = S^2 \cap (\mathbb{R}^2 \times \{0\})$ be the equitorial circle. Let $Y = S^2/\sim$, where we identify $x \sim -x$ for each $x \in E$. Show that Y can be given the structure of a CW-complex, being careful to describe the cells and attaching maps. Use this to compute H_*Y.

3. (25 points) Prove the following simple version of invariance of dimension: if m and n are positive integers such that $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$. Do not invoke invariance of dimension or domain in your proof.

4. (25 points) Show that if a path connected, locally path connected space X has finite fundamental group, then every map $X \to S^1$ is null homotopic. (Hint: use the covering map $\mathbb{R} \to S^1$.)