Comprehensive Exam in Topology
University of Illinois, January 2009

1. (25 points) Let H_* denote singular homology with integer coefficients. Let X be a space, and let

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_k \subseteq \cdots X$$

be a sequence of open subsets of X such that $\bigcup_{k=1}^{\infty} U_k = X$. Let $z \in H_n X$.

Using *only* the definition of singular homology, show that there is a k such that

$$z \in \text{Image}(H_n U_k \to H_n X).$$

2. (25 points)

Show that a (not necessarily connected) space X is simply connected if and only if every pair of continuous maps $f_0, f_1 \colon S^1 \to X$ are homotopic to each other.

3. (25 points)

Let S be a regular hexagon in the plane with vertices $P_1, \ldots, P_6$ (listed counter-clockwise). Let X be the closed subset of the plane enclosed by S. Let Y be the quotient space of X obtained by making the identifications:

- Identify the edge P_1P_2 with the edge P_2P_3.
- Identify the edge P_4P_3 with the edge P_1P_6.

(Note that by "identify edge AB with CD", we mean that each point along AB is identified with the corresponding point along CD, so that in particular, $A \sim C$ and $B \sim D$ in the quotient.)

(a) Describe $\pi_1 Y$ in terms of generators and relations.

(b) Describe $H_1 Y$ in terms of generators and relations.

4. (25 points) Let $X = S^1 \times S^1$, and let $A = \{(a, b) \in S^1 \times S^1 \mid a = b\}$. Compute the relative homology groups $H_*(X, A)$.