Math 525 comprehensive exam
UIUC, August, 2008

1. *(10 points)* Let Y be the connected sum of two Klein bottles. That is, take two Klein bottles K_0 and K_1, remove a disk from each and then connect the two punctured Klein bottles with a cylinder $S^1 \times [0,1]$ along the boundaries of the removed disks via homeomorphisms. Compute $\pi_1(Y)$ and $H_*(Y)$.

2. *(10 points)* Show that a space X is simply-connected if and only if all (continuous) maps $S^1 \to X$ are homotopic.

3.(*10 points*) Let $n \geq 1$ and let x_0, x_1, x_2 be three distinct points in $\mathbb{R}^n$. Let $X = \mathbb{R}^n - \{x_0, x_1, x_2\}$. Compute $\pi_1(X)$ and $H_*(X)$ (your answer will depend upon n).

4.(*20 points, 5 each*) Let Y be a space and G a group acting on Y such that the group homomorphism $G \to \text{Homeo}(Y,Y)$ is injective. Assume furthermore that the action of G satisfies the property:

for each $y \in Y$ there is an open neighborhood U such that
$$g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2 \text{ for all } g_1, g_2 \in G.$$

We let Y/G be the quotient space of Y obtained by identifying orbits ($y \sim y' \Leftrightarrow gy = y'$ for some $g \in G$). We let $p : Y \to Y/G$ be the quotient map.

a. Prove that p is a covering map.

b. Prove that Y is Hausdorff if and only if Y/G is Hausdorff.

c. Prove that if Y is path connected than Y/G is path connected. Give an example showing that the converse is not true (i.e. where Y/G is path connected but Y is not path connected).

d. Prove that if Y is simply-connected than Y/G is semilocally simply-connected.