COMPREHENSIVE EXAM IN TOPOLOGY

AUGUST 2007

1. Let $f : A \to X$ be a map. Recall that a *retraction* of f is a map $g : X \to A$ such that $gf = 1_A$.

(1) Is there a map $f : S^2 \to S^3$ which admits a retraction? If yes, give an example; if not, prove it.

(2) Is there a map $f : S^1 \to \mathbf{R}P^2$ which admits a retraction? If yes, give an example; if not, prove it.

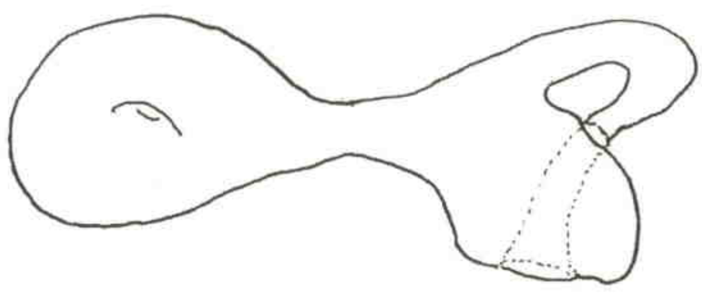
2. Let $P = \mathbf{R}P^2$ be the real projective space of lines through the origin in $\mathbf{R}^3$. Classify the two-sheeted covering spaces of $P \times P$. Justify your answer.

3. Let $n > 1$ be a natural number, and let A be the abelian group $A = \mathbf{Z}/n\mathbf{Z}$. Let $k > 0$ be a natural number. Is there a space X with

$$\begin{aligned} H_0(X;\mathbf{Z}) &= \mathbf{Z} \\ H_k(X;\mathbf{Z}) &\cong A \\ H_p(X;\mathbf{Z}) &= 0 \qquad\qquad p \neq 0, k. \end{aligned}$$

If yes, construct it. If no, explain why not.

4. Let T be the torus, and K the Klein bottle. Let X be the space obtained from T and K by cutting a disk out of each, and then identifying the boundary circles. (See the figure). Calculate $H_*(X;\mathbf{Z})$.



As usual, the neck of the Klein bottle does not actually pass through the side, but must appear to when drawn in $\mathbb{R}^2$ or $\mathbb{R}^3$.