COMPREHENSIVE EXAM IN TOPOLOGY

AUGUST 2006

1. For $g \geq 0$ let M^g be a 2-sphere with g handles, that is, M^g is homeomorphic to the surface of genus g. Suppose that all the handles lie in the northern hemisphere. Let R^g be the space obtained from M^g by removing a single point, say the south pole.

(1) Show that

$$\begin{aligned} H_0(R^g;\mathbb{Z}) &\cong \mathbb{Z} \\ H_1(R^g;\mathbb{Z}) &\cong \mathbb{Z}^{2g} \\ H_{k>1}(R^g;\mathbb{Z}) &= 0. \end{aligned} \tag{1}$$

(2) Show that the image under the Hurewicz homomorphism

$$\pi_1(R^g;p) \to H_1(R^g;\mathbb{Z})$$

of the equator between the north and south poles is zero. p can be any point on the equator.

(3) Calculate $H_*(M^g;\mathbb{Z})$.

2. Prove that, if $n > k$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^k$.

3. Let $I = [0,1]$. Let X be a space, and let p and q be two points of X.

(1) Give an example of a *connected* space X and points p and q such that

$$\pi_1(X;p) \ncong \pi_1(X;q).$$

(2) Show on the other hand that if there is a path $\gamma : I \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$, then there is an isomorphism

$$c_\gamma : \pi_1(X;p) \cong \pi_1(X;q). \tag{2}$$

(3) Show that if X is path-connected and $\pi_1(X;p)$ is an *abelian* group, then the isomorphism (2) does not depend on the choice of γ : if γ' is another path in X with $\gamma'(0) = p$ and $\gamma'(1) = q$, then

$$c_{\gamma'} = c_\gamma : \pi_1(X;p) \longrightarrow \pi_1(X;q).$$

4. Let D be a closed 2-dimensional disk with boundary circle C. Let C' be another circle, and let $f : C \to C'$ be a map of degree n. Let X be the identification space

$$X = \frac{D \coprod C'}{C \ni c \sim f(c) \in C'}.$$

(1) Calculate $\pi_1(X;0)$, where 0 is the image in X of the center of D.

(2) Calculate $H_*(X;\mathbb{Z})$.