Math 520

Comprehensive Exam: Differentiable Manifolds
May 2009

Problem 1
Consider the subset of $\mathbb{R}^4$ satisfying the following two equations

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4 \quad \text{and} \quad x_1 x_2 x_3 = 1.$$

Prove that this is a manifold. What is its dimension?

Problem 2
Consider the following two 1–parameter families of maps

$$\phi_t(x, y, z) = (e^t x, y + t, z + t)$$

and

$$\psi_t(x, y, z) = (e^t x + t, y, z).$$

(a) Prove that ϕ_t defines a flow, but that ψ_t does not.

(b) Find the vector field ξ that generates ϕ_t.

(c) Set $\omega = z\,dx + y^2\,dy + x\,dz$ and compute $L_\xi \omega$.

Problem 3
Let S^2 denote the unit sphere in $\mathbb{R}^3$ defined by

$$x^2 + y^2 + z^2 = 1$$

given the standard orientation. That is, $\{\partial/\partial y\,,\ \partial/\partial z\}$ is positively oriented at $(1, 0, 0)$. Let α be the 2–form defined by

$$\alpha = (3x^2 \cos(y) + e^{xy})\,dx \wedge dy + 17x^3\,dx \wedge dz + (x + yz^5 + \sin(z))\,dy \wedge dz.$$

Compute

$$\int_{S^2} \alpha.$$