Differential Geometry Comprehensive Exam, May 2007

1. a. Compute the exterior derivative of the 2-form

$$\phi = \frac{1}{(x^2+y^2+z^2)^{3/2}}(xdy\wedge dz + ydz\wedge dx + zdx\wedge dy)$$

b. Fact: $\int_{S^2}\phi = 4\pi$ where $S^2 = \{(x,y,z)\,|\,x^2+y^2+z^2=1\}$ oriented with the outward normal. Does this and the result of the computation of part (a) contradict Stokes' theorem? Explain.

c. What is $\int_\Sigma \phi$ where $\Sigma = \{(x,y,z)\,|\,\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1\}$ oriented with outward normal?

2. a. Let $f: V_1 \to W$ and $g: V_2 \to W$ be linear maps of finite dimensional vector spaces. Prove that if f is onto then

$$(f(V_1)\oplus 0) + (0\oplus g(V_2)) + \Delta_W = W \oplus W$$

where Δ_W is the diagonal: $\Delta_W = \{(w_1, w_2) \in W\oplus W \mid w \in W\}$.

b. Suppose $f: N_1 \to M$ and $g: N_2 \to M$ are smooth maps between manifolds. Prove that if f is a submersion then

$$Z = \{(x,y) \in N_1 \times N_2 \mid f(x_1) = g(x_2)\}$$

is a submanifold of $N_1 \times N_2$. Hint: part (a) is relevant.

3. Consider the flow $\phi_t(x,y,z) = (e^t x, e^{2t}y, e^{-2t}z)$ on $\mathbb{R}^3$.
a. Find the vector field X that generates the flow.
b. Compute the Lie derivative $L_X(dx\wedge dy\wedge dz)$.
c. Find a 2-form σ so that $L_X(dx\wedge dy\wedge dz) = d\sigma$.