Differential Geometry Comprehensive Exam, May 2006

1. Let $f : M \to \mathbb{R}$ be a smooth function. What are the conditions on the differential of f that guarantee that $N := f^{-1}(0)$ is an embedded submanifold? Are these conditions necessary?

Suppose $N = f^{-1}(0)$ is a submanifold and X, Y are two vector fields on M that are tangent to N. Prove that the Lie bracket $[X, Y]$ is tangent to N as well.

2. The set

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = 8 - 2y^2 - 2z^2 - 2w^2, x \geq 0\}$$

is a 3-manifold with boundary. Pick an orientation of M and compute the integral

$$\int_M d(xy) \wedge dz \wedge dw.$$

3. Let ∇ be a connection on a manifold M. Prove that the torsion T of the connection defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where X, Y are vector fields, is actually a tensor on M. What do you need to check?

4. ~~Let $f : M \to N$ be a submersion and $g : P \to N$ a smooth map (M, N, P are manifolds). Is the set~~

$$\cancel{M \times_N P = \{(m, p) \in M \times P \mid f(m) = g(p)\}}$$

~~a submanifold of $M \times N$? Explain.~~