Math 520

Comprehensive Exam: Differentiable Manifolds

January 2009

Problem 1

The space of 2×2–matrices with real entries $M_2(\mathbb{R})$ is naturally diffeomorphically to $\mathbb{R}^4$ by the diffeomorphism

$$(x, y, z, w) \mapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

Let

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : xw - zy = 1 \right\}$$

(a) Prove that $SL_2(\mathbb{R})$ is a smooth submanifold of $M_2(\mathbb{R})$.

(b) Prove that the vector field

$$\xi = x\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}$$

on $M_2(\mathbb{R})$ is tangent to $SL_2(\mathbb{R})$.

Problem 2

Recall that a function $f : \mathbb{R}^3 \to \mathbb{R}$ is called **harmonic** if

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = 0$$

and that the **gradient** of f is given by

$$\nabla f = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_3}.$$

Prove that f is harmonic if and only if

$$L_{\nabla f}(dx_1 \wedge dx_2 \wedge dx_3) = 0.$$

Here $L_\xi(\omega)$ is the Lie derivative of the form ω with respect to the vector field ξ.

Problem 3.

Let M be a compact oriented smooth n–manifold with boundary $\partial M = N_1 \sqcup N_2$ given the boundary orientation. Let σ be an $n-1$-form on M. Suppose

$$\int_M d\sigma = 0$$

and

$$\int_{N_1} \sigma = 1$$

What is

$$\int_{N_2} \sigma?$$

Explain.