520

Comprehensive Exam: Differentiable Manifolds

January 2007

Problem 1

Let ω be a closed 2–form on the standard three–sphere $S^3 \subset \mathbb{R}^4$ and let Σ be the equator

$$\Sigma = S^3 \cap (\mathbb{R}^3 \times \{0\}) \subset \mathbb{R}^4$$

given either orientation. What can be said about

$$\int_\Sigma \omega \quad ?$$

Explain.

Problem 2

Consider the flow on $\mathbb{R}^3$ given by

$$\phi_t(x, y, z) = (e^t x, e^{-t} y, z + t)$$

(a) Find the vector field ξ which generates this flow.

(b) Compute $[\xi, \frac{\partial}{\partial z}]$.

(c) Compute $L_\xi(dx \wedge dy)$.

Problem 3

Write $\mathbf{x} = (x_1, ..., x_{n+1}), \mathbf{y} = (y_1, ..., y_{n+1}) \in \mathbb{R}^{n+1}$ and define a bilinear function

$$B(\mathbf{x}, \mathbf{y}) = x_1 y_1 + ... + x_n y_n - x_{n+1} y_{n+1}$$

(a) Prove that

$$X = \{\mathbf{x} \,|\, B(\mathbf{x}, \mathbf{x}) = -1\}$$

is a submanifold of $\mathbb{R}^{n+1}$.

(b) Viewing $X \subset \mathbb{R}^{n+1}$, we can identify each tangent space $T_\mathbf{x} X$ with a subspace of $\mathbb{R}^{n+1}$. For each $\mathbf{x} \in X$, describe the subspace $T_\mathbf{x} X \subset \mathbb{R}^{n+1}$.