COMPREHENSIVE EXAM IN GEOMETRY, January 2006

1. In this problem, ω is a 1-form on $\mathbf{R}^3$ which is not 0 at the origin O.

(a) Suppose there are two smooth functions f and g defined on a neighborhood V of O with $f(O) \neq 0$, such that $\omega = f dg$ on V. Show there is a 1-form θ on a neighborhood of O such that $d\omega = \theta \wedge \omega$.

(b) If $\omega = dz - y dx - dy$, do f and g as in part (a) exist? Prove your claim.

2. In the standard coordinates (x, y) on the plane, consider the vectorfield $V = \partial_x + y^3 \partial_y$.

(a) Compute the integral curve of V through an arbitrary initial point (x_0, y_0).

(b) Find the set of initial points (x_0, y_0) for which the flow along V exists for time $t = 1$.

3. Prove that an n-dimensional manifold is orientable (in the sense of carrying an oriented atlas) if and only if it carries a global nowhere vanishing n-form.

4. Let ∇ be the Riemannian connection on an n-dimensional manifold N. Let M be a hypersurface in N with unit normal field U.

(a) For any vectorfield X tangent to M, set $S(X) = \nabla_X U$. Show $S(X)$ is a vectorfield tangent to M.

(b) Show S determines a linear transformation $S_p : T_p M \to T_p M$, for any $p \in M$.