DIFFERENTIAL GEOMETRY COMPREHENSIVE EXAM
AUGUST 2006.

1. Let M be the set of (straight, nondegenerate) circles in the 2-sphere $S^2 \subseteq \mathbb{R}^3$. Describe a natural topology on M. Show that M is a manifold. What is the dimension of M? Is M orientable?

2. Is there an immersion of the 2-sphere S^2 to $S^1 \times \mathbb{R}$?

3. Give a definition of the de Rham cohomology, $H^p_{dR}(M)$, of a smooth manifold M. Prove that if M is a compact orientable manifold (without boundary) of dimension n, then $H^n_{dR}(M) \neq 0$.

4. (a) Give a definition of a connection on a vector bundle over a manifold M.
(b) Let M be a smooth manifold. Is there a connection ∇ on TM such that $\nabla_X Y = \nabla_Y X$ for all vector fields X and Y on M?
(c) Is there a connection on TM such that $\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z$ for all vector fields X, Y and Z on M?