Comprehensive Exam: Math 518

January 2018

Problem 1 (30 points) Given the following objects

- $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$
- $Y = 2x\frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$
- $\alpha = ydx + dz \in \Omega^1(\mathbb{R}^3)$
- $f\colon \mathbb{R}^3 \to \mathbb{R}$, given by $f(x,y,z) = x^2 + 2z$
- $F\colon \mathbb{R}^2 \to \mathbb{R}^3$, given by $F(u,v) = (e^u, e^u \sin v, u^2 + v^2)$
- $\gamma\colon [0, 2\pi] \to \mathbb{R}^3$, given by $\gamma(t) = (\sin t, \cos t, \sin t)$

compute the following quantities:

a. the 1-parameter group of diffeomorphisms (flow) $\phi_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ that corresponds to the vector field X

b. $XY(f)$

c. $\alpha \wedge d\alpha$

d. $\mathcal{L}_Y \alpha$

e. $F^* d\alpha$

f. $\int_\gamma \alpha$.

Problem 2 (15 points)

a. Give an example of two smooth maps G and H such that: (i) H has critical points, (ii) the composition $G \circ H$ is defined, and (iii) the map $G \circ H$ has no critical points.

b. Find all values of k for which the set

$$M_k = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_1^3 - x_2^2 + x_3 x_4 = k\}$$

is an embedded submanifold of $\mathbb{R}^4$. Are the sets M_k compact?

Problem 3 (15 points) Consider the following 2–form on $\mathbb{R}^3$

$$\omega = \frac{z^2}{4}\, dy \wedge dz - y\, dx \wedge dz + x \sin(y^3)\, dx \wedge dy.$$

For $R > 0$, let $S^2(R)$ denote the unit sphere in $\mathbb{R}^3$ defined by

$$x^2 + y^2 + z^2 = R^2$$

and equipped with its standard orientation as the boundary of $\bar{B}^3(R)$, the closed ball of radius R. Compute

$$\int_{S^2(2)} \omega - \int_{S^2(1)} (\omega + df)$$

where f is the function from Problem 1.