Comprehensive exam
Math 518, January 2019.

Name:

(1) Let M be a connected smooth manifold and let $p, q \in M$. Prove that there is a diffeomorphism $F : M \longrightarrow M$ such that $F(p) = q$.

(2) In $\mathbb{R}^3$, with coordinates (x, y, z), let

$$V = z\,\partial_x - z\,\partial_y, \text{ and}$$
$$\omega = x\,dy \wedge dz + x\,dx \wedge dz + z\,dx \wedge dy.$$

Compute the following:
(a) $\mathrm{i}_V\omega$, (b) $d\omega$, (c) $\mathcal{L}_V\omega$,
(d) $\Phi^*\omega$ where $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the map $(s,t) \mapsto (s+t, s-t, t)$.

(3) In $\mathbb{R}^3$, with coordinates (x, y, z), let $V, W \in \mathcal{C}^\infty(\mathbb{R}^3, T\mathbb{R}^3)$ be the vector fields given by

$$V = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}, \quad W = \frac{\partial}{\partial y}.$$

Let $E \subseteq T\mathbb{R}^3$ be the sub-bundle spanned by $\{V, W\}$.
(a) Is there an integral submanifold of E through an arbitrary point of $\mathbb{R}^3$?
(b) Find an integral submanifold of E passing through the origin.

(4) Let $\mathcal{U} \subseteq \mathbb{R}^3$ be an open subset with smooth boundary Σ, each oriented with the induced orientation. Suppose $\int_{\mathcal{U}} dx\, dy\, dz = \mathrm{Vol}(\mathcal{U}) = V$, evaluate

$$\int_\Sigma e^{\cos(x+y^2)}\, dx \wedge dy + (x^2 z + xz + z^3)\, dx \wedge dz + 2z\, dy \wedge dx$$