Comprehensive exam

Name:

(1) Let $M$ be a connected smooth manifold and let $p, q \in M$. Prove that there is a diffeomorphism $F : M \rightarrow M$ such that $F(p) = q$.

(2) In $\mathbb{R}^3$, with coordinates $(x, y, z)$, let

$$V = z \partial_x - z \partial_y,$$

and

$$\omega = x \, dy \wedge dz + x \, dx \wedge dz + z \, dx \wedge dy.$$  

Compute the following:
(a) $i_V \omega$,  
(b) $d\omega$,  
(c) $\mathcal{L}_V \omega$,  
(d) $\Phi^* \omega$ where $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the map $(s, t) \mapsto (s + t, s - t, t)$.

(3) In $\mathbb{R}^3$, with coordinates $(x, y, z)$, let $V, W \in C^\infty(\mathbb{R}^3, T\mathbb{R}^3)$ be the vector fields given by

$$V = \frac{\partial}{\partial x} + yz \frac{\partial}{\partial z}, \quad W = \frac{\partial}{\partial y}.$$  

Let $E \subseteq T\mathbb{R}^3$ be the sub-bundle spanned by $\{V, W\}$.
(a) Is there an integral submanifold of $E$ through an arbitrary point of $\mathbb{R}^3$?
(b) Find an integral submanifold of $E$ passing through the origin.

(4) Let $U \subseteq \mathbb{R}^3$ be an open subset with smooth boundary $\Sigma$, each oriented with the induced orientation. Suppose $\int_U dx \, dy \, dz = \text{Vol}(U) = V$, evaluate

$$\int_{\Sigma} e^{\cos(x^2 + y^2)} \, dx \wedge dy + (x^2 z + xz + z^3) \, dx \wedge dz + 2z \, dy \wedge dx.$$