Comprehensive exam
Math 518, January 2017.

(1) Consider the smooth vector field on $\mathbb{R}^2$, $V = -2x\partial_x + 2y\partial_y$. Find the 1-parameter group of diffeomorphisms corresponding to V. That is, compute the flow of V.

(2) In $\mathbb{R}^3$, with coordinates (x, y, z), let

$$V = xy\ \partial_z - \ \partial_y, \text{ and}$$
$$\omega = z\ dx \wedge dy + \ dy \wedge dz$$

Compute the following:
(a) $i_V\omega$, (b) $d\omega$, (c) $\mathcal{L}_V\omega$,
(d) $\Phi^*\omega$ where $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the map $(s, t) \mapsto (s + t, s, t)$.

(3) Consider the pair of vector fields $V = \partial_y$ and $W = y\partial_x - \partial_z$ on $\mathbb{R}^3$, where we use the standard coordinates (x, y, z). Is it possible to find a 2-dimensional submanifold of $\mathbb{R}^3$ with the property that both V and W are tangent to it at all points? If so, construct one; if not, why not?

(4) Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$

$$\alpha = -\frac{y}{x^2 + y^2}\ dx + \frac{x}{x^2 + y^2}\ dy.$$

Prove that $d\alpha = 0$ and hence that the integral of $d\alpha$ over the unit disk

$$D := \{(x, y) \mid x^2 + y^2 \leq 1\}$$

is 0. Show that the integral of α over the unit circle

$$S^1 := \{(x, y) \mid x^2 + y^2 = 1\}$$

is **not** zero. Does this contradict Stokes theorem? Explain.