Comprehensive Exam

Math 518, January 2016

(1) Consider the following vector field on $\mathbb{R}^4$,

$$X = \cos x_4 \frac{\partial}{\partial x_3} + \sin x_2 \frac{\partial}{\partial x_1}.$$

Compute the Lie derivative

$$\mathcal{L}_X(x_1 dx_1 \wedge dx_2 + x_3 dx_3 \wedge dx_4).$$

(2) Let M be an orientable manifold and let N be an embedded submanifold of M of codimension one. Suppose there exists a vector field $X \in \Gamma(TM)$ such that $X(n) \notin T_n N$ for all $n \in N$. Prove that N is orientable.

(3) Consider the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^5.$$

Prove that

$$M = \{x \in S^{n-1} \mid f(x) = 0\}$$

is an embedded submanifold of the unit sphere S^{n-1}.

(4) Suppose that $\alpha \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ is a closed form and that $\int_{S^{n-1}} \alpha \neq 0$. Prove that there **does not** exist a form $\beta \in \Omega^{n-1}(\mathbb{R}^n)$ such that $\beta|_{\mathbb{R}^n \setminus \{0\}} = \alpha$.