# Comprehensive exam

Math 518, January, 2015

1. Consider the function $F: \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2, x_3^2 + x_4^2).$$

   (a) Prove that $(1,1)$ is a regular value and hence $X = F^{-1}(1,1)$ is a submanifold of $\mathbb{R}^4$. What is its dimension?

   Let $i: X \to \mathbb{R}^4$ denote the inclusion and let

$$\alpha = -x_2\, dx_1 + x_1\, dx_2 - x_4\, dx_3 + x_3\, dx_4.$$

   (b) Prove that $i^*\alpha$ is closed; that is, prove $d(i^*\alpha) = 0$.

   (c) Prove that $i^*\alpha$ is not exact; that is, prove that $i^*\alpha \neq df$ for any smooth function $f: X \to \mathbb{R}$. *Hint:* Consider $X \cap V$ where $V$ is the plane $x_3 = x_4 = 0$.

2. Suppose $M$ is a smooth $n$–manifold oriented by a nowhere vanishing $n$–form $\omega$. The *$\omega$–divergence* of a smooth vector field $\xi$ on $M$ is defined to be the function $\operatorname{div}(\xi)$ such that

$$\mathcal{L}_\xi(\omega) = \operatorname{div}(\xi)\omega.$$

   where $\mathcal{L}_\xi(\omega)$ is the Lie derivative of $\omega$ with respect to $\xi$.

   (a) Prove that the equation above well-defines $\operatorname{div}(\xi)$ as a smooth function.

   (b) Prove that on $\mathbb{R}^3$ with volume form $\omega = dx \wedge dy \wedge dz$, the $\omega$–divergence of a smooth vector field $\xi = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ is given by

$$\operatorname{div}(\xi) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

   *Emergency backup*: If you cannot prove this for an arbitrary vector field, prove it for the vector field generated by the flow

$$\varphi_t(x, y, z) = (\cos(t)x + \sin(t)y, -\sin(t)x + \cos(t)y, e^t z).$$

3. Let $S_R^2$ be the sphere of radius $R$ centered at the origin in $\mathbb{R}^3$. Let $i: S_R^2 \to \mathbb{R}^3$ be inclusion and set

$$\beta = x\, dy \wedge dz - y\, dx \wedge dz + \sin(x)\, dx \wedge dy.$$

   Choose an orientation, and evaluate $\displaystyle\int_{S_R^2} i^*\beta$.