- Math 518 - Fall 2013 -
Differentiable Manifolds I
Comprehensive Exam
(January 2014)

1. On $\mathbb{R}^3$ with coordinates (x, y, z) consider the vector fields:

$$V = \frac{\partial}{\partial y}, \quad W = e^{-y}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$$

Is there any 2-dimensional submanifold $N \hookrightarrow \mathbb{R}^3$ such that:

$$T_x N = \langle V|_x, W|_x \rangle, \quad \forall x \in N?$$

Justify your answer.

2. In $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with angle coordinates $(\theta_1, \theta_2, \theta_3)$, let $X = \sin\theta_1 \frac{\partial}{\partial\theta_3}$ and $\omega = \cos\theta_2 \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_3 + \sin\theta_1 \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3$. Compute the following:

(a) $i_X\omega$; (b) $\mathrm{d}\omega$; (c) $\mathcal{L}_X\omega$.

3. Let M be a 8-dimensional compact manifold without boundary. Let $\omega \in \Omega^4(M)$ be a differential form and assume that $\omega \wedge \omega$ is a volume form. Show that there is no 3-form $\alpha \in \Omega^3(M)$ such that $\omega = \mathrm{d}\alpha$.
Hint: Observe that if $\omega = \mathrm{d}\alpha$ then $\omega \wedge \omega = \mathrm{d}(\alpha \wedge \omega)$.

4. Let $n \geq 1$. Show that:

$$N = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{RP}^n : x_0^3 + \cdots + x_n^3 = 0\}$$

is a submanifold of the real projective space $\mathbb{RP}^n$ and compute its dimension.