518

Differential Geometry Comprehensive Exam, January 2012

1. Let $D \subset \mathbb{R}^3$ be a compact domain with smooth boundary Σ, which is given the induced orientation. Suppose $\int_D 1\, dx\, dy\, dz = V$ (i.e., the volume of D is V). What is

$$\int_\Sigma e^{\cos(x+y^2)}\, dx \wedge dy + (x^2 z + xz + z^3)\, dx \wedge dz + 2z\, dy \wedge dx?$$

Explain.

Solution. Set

$$\omega = e^{\cos(x+y^2)}\, dx \wedge dy + (x^2 z + xz + z^3)\, dx \wedge dz + 2z\, dy \wedge dx.$$

Then

$$d\omega = 2dz \wedge dy \wedge dx = -2dx \wedge dy \wedge dz.$$

By Stokes' Theorem

$$\int_\Sigma \omega = \int_D d\omega = \int_D -2dx \wedge dy \wedge dz = -2V.$$

2. a. Prove that 6 is a regular value of the function $f : \mathbb{R}^3 \to \mathbb{R}$ which is given by $f(x, y, z) = 2x^2 + 3y^2 + z^2$.

Solution. We first observe that

$$df_{(x,y,z)} = 4xdx + 6ydy + 2zdz = 0 \Leftrightarrow x = y = z = 0.$$

Then since $f(0, 0, 0) = 0 \neq 6$, the derivative $df_{(x,y,z)}$ has rank one for every $(x, y, z) \in f^{-1}(6)$ and so 6 is a regular value.

b. Explain why $f^{-1}(6)$ is a submanifold of $\mathbb{R}^3$. That is, state the relevant theorem.

Solution. The Regular Value Theorem states that the preimage of a regular value is a submanifold. Since 6 is a regular value of f, it follows that $f^{-1}(6)$ is a submanifold.

c. If $f : X \to Y$ is a smooth map of manifolds and $y \in Y$ is a regular value, prove that $T_x(f^{-1}(y)) = \ker df_x$ for any $x \in f^{-1}(y)$.

Solution. The RVT further states that $f^{-1}(y)$ has dimension equal to $\dim(X) - \dim(Y)$. Therefore, it suffices to show that $T_x(f^{-1}(y)) \subset \ker df_x$ for any $x \in f^{-1}(y)$. For this we observe that given a tangent vector $v \in T_x(f^{-1}(y))$ there is a curve $\gamma : (\epsilon, \epsilon) \to f^{-1}(y)$ with $\gamma(0) = x$ and $\dot\gamma(0) = v$. Then the chain rule implies

$$df_x(v) = df_x(\dot\gamma(0)) = df_x \circ d\gamma_0(\mathbf{1}) = d(f \circ \gamma)_0(\mathbf{1}) = 0$$

where the last equality comes from the fact that $f \circ \gamma$ is constant equal to y.

3. a. Prove that $\phi_t(x, y, z) = (x, e^{2t}y, e^{-3t}z)$ defines a flow on $\mathbb{R}^3$.

Solution. We calculate

$$\phi_0(x, y, z) = (x, e^0 y, e^0 z) = (x, y, z)$$

and

$$\begin{aligned} \phi_{s+t}(x, y, z) &= (x, e^{2(s+t)}y, e^{-3(s+t)}z) \\ &= (x, e^{2s}(e^{2t}y), e^{-3s}(e^{-3t}z)) \\ &= \phi_s(x, e^{2t}y, e^{-3t}z) \\ &= \phi_s(\phi_t(x, y, z)) \\ &= \phi_s \circ \phi_t(x, y, z). \end{aligned}$$

So ϕ_0 is the identity and $\phi_{s+t} = \phi_s \circ \phi_t$ and hence ϕ is a flow.

b. Find the vector field X that generates the flow.

Solution. We differentiate to find the vector field

$$X = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x, y, z) = (0, 2e^{2\cdot 0}y, -3e^{-3\cdot 0}z) = (0, 2y, -3z).$$