Comprehensive Exam: Differentiable Manifolds

January 2010

Problem 1

Consider the function $F\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x, y, z) = x^2y + yz + z^2.$$

(a) Find all the regular points of the map F.

(b) Show that the equation $F(x, y, z) = 1$ defines a smooth submanifold M of $\mathbb{R}^3$.

(c) Let $\mathbf{x} = (x_0, y_0, z_0)$ be a point on M. Determine the equation(s) which define the tangent space $T_{\mathbf{x}}M$ as a linear subspace of $T_{\mathbf{x}}\mathbb{R}^3$.

(d) Let π be the projection map from M to the xz-plane. Find all the points of M at which π is not regular.

Problem 2

(a) Define a vector field V on $\mathbb{R}^2$ whose time one flow maps the origin to the point $(a, b) \in \mathbb{R}^2$.

(b) Let p and q be points of a smooth connected manifold M. Prove that there is a diffeomorphism $F\colon M \to M$ such that $F(p) = q$.

Problem 3

Let M be a compact, oriented and connected n-dimensional manifold with boundary, ∂M. Let $i\colon \partial M \to M$ be inclusion. For a k-form α on M and an $(n-k-1)$-form β such that $i^*\beta = 0$, show that

$$\int_M d\alpha \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge d\beta.$$