Math 518

Differential Geometry Comprehensive Exam, August 2012

1. Let σ be a closed form on a manifold M and X a vector field on M. Prove that the Lie derivative of σ with respect to X is exact.

2. Consider the closed ball

$$D := \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

with the standard orientation given by the form $\mu := dx_1 \wedge \ldots \wedge dx_n$. The form μ induces an orientation on the boundary ∂D. Write down the form ν on the boundary ∂D that represents this orientation. Prove that ν is indeed nowhere zero..

3. Show that an m-dimensional oriented closed manifold (i.e., compact and without boundary) carries a closed m-form which is not exact.

4. Consider the flow in $\mathbb{R}^3$ given by

$$\varphi_t(x, y, z) = (e^t x, y + t, e^{-2t} z).$$

a. Compute the vector field X which generates the flow.

b. Compute the Lie derivative $L_X(dx \wedge dy)$.