Comprehensive Exam: Differentiable Manifolds
August 2011

Problem 1 (15 points)
Show that the subset

$$\{(x, y) \in \mathbb{R}^2 \mid x^3 + xy + y^3 = 1\}$$

is a submanifold of $\mathbb{R}^2$.

Problem 2 (35 points)

(a) Let x, y, z be the standard coordinate functions on $\mathbb{R}^3$. Consider the vector fields

$$\begin{aligned} X &= x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}, \\ Y &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \end{aligned}$$

and the 2-form

$$\omega = (x - y)dx \wedge dy + (x + y + z)dy \wedge dz$$

on $\mathbb{R}^3$. Compute the following quantities.

(i) The time-t flow, ϕ_t, of the vector field X.

(ii) The push forward map $(\phi_1)_* : T_{(x,y,z)}\mathbb{R}^3 \to T_{\phi_1(x,y,z)}\mathbb{R}^3$.

(iii) The Lie bracket $[X, Y]$.

(iv) The exterior derivative $d\omega$.

(b) (i) For a smooth vector field Y and a smooth k-form ω on a smooth manifold M define the Lie derivative $\mathcal{L}_Y\omega$

(ii) For Y and ω as in part (a), compute $\mathcal{L}_Y\omega$.

Problem 3 (30 points)

(i) Let W be a compact oriented manifold of dimension $k+1$ with nonempty boundary $\partial W = M$. Let $F: M \to N$ be a smooth map and ω a smooth k-form on N such that $d\omega = 0$. Prove that if F can be extended to a smooth map $\widetilde{F}: W \to N$, then

$$\int_M F^*\omega = 0.$$

(ii) For $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ consider the smooth map $F\colon S^1 \to S^1$ defined by

$$F(x,y) = (-x,-y).$$

Prove that F cannot be extended to a smooth map $\widetilde{F}\colon B^2 \to S^1$ where $B^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$.

Hint. Consider part (i) and the one-form ω on S^1 defined as the restriction of

$$\left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$