MATH 518 (formerly MATH 520)

comprehensive exam

August 2009

1. Let $\mathbb{R}^2$ have coordinates (u, v) and let $\mathbb{R}^3$ have coordinates (x, y, z). In $\mathbb{R}^2$, let $U = (0, \pi) \times (0, 2\pi)$, and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(u, v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$$

(a) Compute $f^*(x)$, $f^*(y)$, $f^*(z)$, $f^*(dx)$, $f^*(dy)$ and $f^*(dz)$.
(b) Consider the differential form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $\mathbb{R}^3$. Compute

$$\int_U f^*(\omega).$$

2. Let M and N be compact oriented n–manifolds-without-boundary and ω an n–form on N such that

$$\int_N \omega = 1.$$

Recall that the degree of a smooth map $f : M \to N$ can be defined as

$$\deg(f) = \int_M f^*(\omega).$$

Prove that $\deg(f)$ depend only on the homotopy class of f; that is, if f is homotopic to g, then $\deg(f) = \deg(g)$.

3. Let M be a smooth noncompact manifold-without-boundary and $K \subset M$ any compact subset. Prove that there exists a compact manifold with boundary N such that $K \subset N \subset M$. (Hint: you may assume the existence of a proper smooth function $f : M \to \mathbb{R}$).