**Algebraic Geometry 511 comprehensive exam** **Spring, 2017**

## 1 Grassmannian

1. Give the definition of regular and rational maps. Define what it means for a variety $X$ to be rational.

2. Prove that the Grassmannian $\mathbb{G}(k,n)$ is rational.

3. What is the dimension of $\mathbb{G}(k,n)$? Justify.

## 2 Segre embedding

1. Give the definition of the Segre embedding $\iota : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Write down $\iota$ explicitly in terms of the homogeneous coordinates of the $\mathbb{P}^1$ factors.

2. Let $Y = \iota(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ (so that we have an isomorphism $Y \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$ which we also denote by $\iota$). Give a homogeneous polynomial $F$ on $\mathbb{P}^3$ so that $Y = V(F)$.

3. Let $\iota^{-1} : Y \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$ denote the inverse isomorphism. Then there is a natural morphism $f : Y \to \mathbb{P}^1$ given by $f = \pi_1 \circ \iota^{-1}$, where $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is projection onto the first factor. Now *forgetting the construction of $Y$ by the Segre embedding* and *only using the description of $Y$ as* $V(F) \subset \mathbb{P}^3$, prove that $f$ is a regular map.

## 3 Projections.

Let $p = (1,0,0) \in \mathbb{P}^2$. This question concerns the projection from $p$, which we write as a rational map $f : \mathbb{P}^2 - - \to \mathbb{P}^1$.

1. Give a formula for $f$ in coordinates. What is the largest open subset $U \subset \mathbb{P}^2$ on which $f$ is regular?

2. Let $\pi : X \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $p = (1,0,0)$. Identify $X$ with an explicit projective subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$. What is $\pi^{-1}(U) \subset X$?

3. Show that $f \circ \pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathbb{P}^1$ extends to a morphism $\tilde{f} : X \to \mathbb{P}^1$. What are the fibers of $\tilde{f}$?

4. Use your answer to question (3) to prove that $X$ is irreducible.

## 4 Smoothness

1. Give two definitions of a nonsingular point $p$ of a variety $X$. First, define smoothness in terms of the local ring of $X$ at $p$. Next, define smoothness in terms of the tangent space of $X$ at $p$ and $dim(X)$. Finally, give a formula for the tangent space of $X$ at $p$ in terms of an affine neighborhood $U$ of $p$ in $X$ and a represention of $U$ as a closed subset $U = V(f_1, \dots, f_k) \subset \mathbb{A}^N$.

2. For the surface $S = V(xy^2 - zw^2) \subseteq \mathbb{P}^3$ find the singular locus.

3. For the smooth point $p = (1:1:1:1)$ find the tangent space to the curve $V(xy^2 - zw^2, x^3 + y^3 - z^3 - w^3)$ as a linear subspace of $\mathbb{P}^3$. Give your answer as $V(L_1, L_2)$ where $L_i$ are homogeneous linear forms.

## 5 Normalization

1. Define what it means for an integral domain to be normal and for an affine variety to be normal.

2. The curve $V(zy^3 - x^4) \subseteq \mathbb{P}^2$ is not normal. Find its normalization and justify your answer.