Math 511 Comp Exam

Problem 1. Let X be an irreducible variety in $\mathbb{A}^n_k$ with k algebraically closed.

(1) Give the definition of the coordinate ring $A(X)$ of X.
(2) Give the definition of the field of rational functions $K(X)$ of X.
(3) Define the dimension of X in terms of $K(X)$ and k.
(4) Define what it means for X to be normal.

Problem 2. Map $\mathbb{P}^1$ to $\mathbb{P}^3$ via

$$[s:t] \xrightarrow{\phi} [s^3 : s^2t : st^2 : t^3], \text{ with } [X_0 : X_1 : X_2 : X_3] \text{ coordinates on } \mathbb{P}^3.$$

(1) Find equations defining the ideal of the image of ϕ. (Hint: there are three of them; you need not prove they are everything)
(2) Give the definition of a regular map. Prove that ϕ is regular.
(3) Map $\mathbb{P}^2$ to $\mathbb{P}^4$ via $[s:t:u] \xrightarrow{\mu} [s^2 : st : su : tu : t^2]$. Is μ a regular map? If not, why not?
(4) Give the definition of a rational map. Is ϕ rational? Is μ rational?

Problem 3. Smoothness

(1) Give two definitions of a nonsingular point p of a variety X. First, define smoothness in terms of the local ring at p. Next, define smoothness in terms of the tangent space at p, and $dim(X)$.
(2) For the curve $C = V(x^2y^2 + x^2z^2 + y^2z^2) \subseteq \mathbb{P}^2$, find the singular points (hint: work on a distinguished affine patch and use the symmetry. Remember the point has to lie on the curve!).
(3) Recall that the tangent cone at a singular point q consists of lines thru q which are limiting positions of secant lines. Give the equation for the tangent cone to C at $(0,0,1)$.

Problem 4.

(1) Prove that the polynomial ring $k[t]$ is its own normalization. [Hint: assume a rational function, as a fraction in lowest terms, satisfies a monic polynomial equation, and derive a contradiction unless the denominator is already a scalar.]
(2) Use part (1) to prove that the normalization of the curve

$$Z(y^2 - x^3) \subset \mathbb{A}^2$$

is isomorphic to $\mathbb{A}^1$.

Problem 5. Let $C = Z(y^4 - x^5) \subset \mathbb{A}^2$. Is the proper transform of this curve in the blow-up $\mathrm{Bl}_0\, \mathbb{A}^2$ nonsingular? Justify your answer.