MATH 501

Comprehensive Exam – May 2012

1. The diagram of modules and module homomorphisms shown has exact rows and commutative squares. Also α is surjective and β and δ are injective. Prove that γ is injective.

$$\begin{array}{ccccccc} A & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & C & \xrightarrow{\nu} & D \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow \\ A' & \xrightarrow{\lambda'} & B' & \xrightarrow{\mu'} & C' & \xrightarrow{\nu'} & D' \end{array}$$

2.

(a) A module is called *noetherian* if it has no infinite ascending chains of submodules. Prove that a module is noetherian if and only if every submodule is finitely generated.

(b) Prove that a finitely generated module over a finitely generated commutative ring with identity is noetherian.

(c) Let M be a noetherian module and let N be a submodule such that M and M/N are isomorphic. Prove that $N = 0$.

3.

(a) State Wedderburn's Theorem on semisimple artinian rings.

(b) Let A be a finite abelian group. Prove that the group algebra $\mathbb{Q}A$ is a direct sum of algebraic number fields $F_1, F_2, \ldots, F_k$ and that $|A| = \sum_{i=1}^{k} (F_i : \mathbb{Q})$. (Here $\mathbb{Q}$ is the field of rational numbers).

(c) Let A be a cyclic group with prime order p. Identify the irreducible $\mathbb{Q}$-representations of A. What do these tell you about the structure of $\mathbb{Q}A$?

4.

(a) Let M be an R-module where R is a ring with identity. Prove that M is projective if and only if it is a direct summand of a free R-module.

(b) Prove that a module M is projective if every exact sequence $0 \to A \to B \to M \to 0$ splits.

5.

(a) Let M be a left R-module where R is an arbitrary ring. Prove that the functor $- \otimes_R M$ is right exact, but not necessarily left exact.

(b) Let R be the polynomial ring $K[x_1, x_2, x_3]$, where K is a field, and let I denote the ideal generated by x_1, x_2, x_3. Prove that the R-module $I \otimes_R I$ can be generated by 9 elements and no fewer than 9. [Hint: I/I^2 is a K-vector space].