DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE
EXAMIN^TION
May **2011**

Problem 1 **a.** (10 points) Let A be a (possibly non commutative) ring with 1 and with the property that any element a of A is either invertible or nilpotent ($a^N = 0$ for some $N \geq 0$). Show that A is a local ring, i.e., A has a unique maximal ideal. (Hint: show that if x is nilpotent, then $1 - x$ is invertible.)

b. (10 points) Let M be an R-module such that $\mathrm{End}_R(M)$ $(= \{f \mid f\colon M \to M \text{ is } R\text{-linear}\})$ is local. Prove that M is indecomposable (i.e., non-zero and not the direct sum of proper nonzero submodules).

Problem 2 Let A be a commutative ring with 1 and let M be an A-module. M is called divisible if for all $a \neq 0 \in A$ the multiplication map $M \xrightarrow{a} M$ is surjective.

a. (10 points) Let A be an integral domain. Show that every injective module over A is divisible.

b. (10 points) Show that any divisible module over a PID is injective. Deduce that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective over $\mathbb{Z}$.

Problem 3

a. (8 points) Let V be a finite dimensional vector space over a field k. Let $T \in \mathrm{End}_k(V)$ and let $f(x) = (x-\alpha)^d$, $\alpha \in k$, $d \geq 1$ be its minimal polynomial. Assume that $V \simeq k[x]/(f(x))$. Show that V has a basis over k such that the matrix for T with respect to this basis is given by

$$\begin{pmatrix} \alpha & 0 & \dots & \dots & 0 \\ 1 & \alpha & 0 & \dots & \vdots \\ 0 & 1 & \alpha & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 & \alpha \end{pmatrix} \tag{1}$$

b. (4 points) Let, as above, T be a linear operator on a finite dimensional vector space, and suppose it has characteristic polynomial $f(x) = (x - \alpha_1)^{d_1}(x - \alpha_2)^{d_2} \dots (x - \alpha_k)^{d_k}$. Write down at least one possible form for the minimal polynomial for T.

c. (8 points) With T as in part **b.** show that there exists a basis for V such that matrix of T consists of blocks, each one of which is of the form (1) in part **a.**

Problem 4 Let $f\colon A \to B$ be a ring homomorphism of commutative rings. This makes B into an A-module, using f. Suppose B is A-flat.

a. (10 points) Let $I \subset J$ be two ideals of A. Prove that $J/I \otimes_A B \xrightarrow{\sim} JB/IB$, where $JB = f(J)B$, and $IB = f(I)B$).

b. (10 points) Moreover, suppose that B has the property that for any A-module N $N \otimes_A B = 0$ implies that $N = 0$. Then show that for every ideal $I \subset A$ we have $f^{-1}(IB) = I$.

Problem 5 (20 points) Let C be a cyclic group of order a prime number p. Prove that $\mathbb{Q}[C]$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}(\epsilon)$, where ϵ is a p-th root of unity, and $\mathbb{Q}[C]$ is the rational group algebra of C.