DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
MAY 2010

Problem 1 (25 points) Let A be a commutative ring with 1 and let M be an A-module. M is called divisible if for all $a \neq 0 \in A$ the multiplication map $M \xrightarrow{a} M$ is surjective.

a. Let A be an integral domain. Show that every injective module over A is divisible.

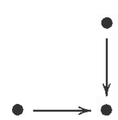
b. Show that any divisible module over a PID is injective. Deduce that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective over $\mathbb{Z}$.

Problem 2 (25 points) Let $f\colon A \to B$ be a ring homomorphism of commutative rings. This makes B into an A-module, using f. Suppose B is A-flat.

a. Let $I \subset J$ be two ideals of A. Prove that $J/I \otimes_A B \xrightarrow{\sim} JB/IB$, where $JB = f(J)B$, and $IB = f(I)B$).

b. Moreover, suppose that B has the following property: if N is an A-module then $N \otimes_A B = 0$ implies that $N = 0$. Show then that for every ideal $I \subset A$ we have $f^{-1}(IB) = I$.

Problem 3 (25 points) Consider a category I with three objects and (besides the identities) morphisms given as follows:



a. Let C be a category and $\mathcal{F}\colon \mathsf{I} \to \mathsf{C}$ a covariant functor. Explain what a limit of $\mathcal{F}$ in C is.

b. Show that if $\mathsf{C} = R$-Mod, for some ring R, then the limit of a covariant functor $\mathcal{F}\colon \mathsf{I} \to \mathsf{C}$ as above always exists. Give an explicit module and morphisms that represents the limit. (Hint: it might be useful to think about the kernel of an appropriate morphism.)

See next page for the fourth problem!

Problem 4 (25 points) (Permutation representations). Let X be a finite set with an action of a finite group G. Let $\mathcal{F}(X)$ be the vector space (over the complex numbers $\mathbb{C}$) with basis $e_x, x \in X$, and define and action of G on $\mathcal{F}(X)$ by $g \cdot e_x = e_{gx}$. $\mathcal{F}(X)$ is called the permutation representation of X.

(1) Show that the character of the permutation representation counts the elements fixed by g:
$$\chi_{\mathcal{F}(X)}(g) = \#\{x \in X \mid gx = x\}$$

(2) Consider the group S_3 of permutations of 3 letters. Let S_3 act on $V = \mathbb{C}^3$ by permuting the elements of a basis. Find the character of V. Is V irreducible?

(3) Give the character table of S_3, i.e., the values of the irreducible characters on all conjugacy classes. Explain.

(4) Determine the decomposition of the permutation representation $V = \mathbb{C}^3$ over S_3 in irreducibles: write V as a direct sum $n_1 V_1 \oplus n_2 V_2 \oplus n_3 V_3$, and find n_1, n_2, n_3.