Math 501

Comprehensive Exam

(Answer all five questions)

1. Let R be a ring with an identity element.

(a) If M is any left R-module, prove that there is a free module F such that $M \simeq^R F/S$ for some submodule S of F.

(b) Prove that every left R-module M has a *free R-resolution*, i.e., an infinite exact sequence of left R-modules and R-module homomorphisms

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

in which each F_i is a free R-module.

2. Let R be a commutative ring with identity and let M, N be R-modules.

(a) If M can be generated as a module by n elements, where n is finite, prove that $\mathrm{Hom}_R(M, N)$ is isomorphic with a submodule of a direct sum of n copies of N.

(b) If both the modules M and N have composition series (of finite length), prove that $\mathrm{Hom}_R(M, N)$ also has a composition series.

3.

(a) Let G be a finite group. If $n_1, n_2, \ldots, n_k$ are the degrees of the irreducible representations of G over the complex field $\mathbb{C}$, prove that

$$|G| = n_1^2 + n_2^2 + \cdots + n_k^2.$$

(b) Let G be a group of order 3. Find three irreducible representations of G over $\mathbb{C}$ and use the group algebra to show that the group has no further irreducible representations over $\mathbb{C}$.

(c) With G as in 3(b), write $\mathbb{C}G$ as the sum of three ideals which give rise to the three irreducible representations of G.

4.

(a) Let ${}_S L_R, M_R, N_R$ be modules over rings R, S as shown. Prove that

$$L \otimes_R (M \oplus N) \simeq (L \otimes_R M) \oplus (L \otimes_R N),$$

where the isomorphism is of left S-modules.

(b) Simplify as far as possible

$$((\mathbb{R} \oplus \mathbb{Z}_6) \otimes (\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8)) \otimes (\mathbb{Z}_{14} \oplus \mathbb{Z}),$$

where all tensor products are over $\mathbb{Z}$ and $\mathbb{R}$ is the additive group of real numbers.

5.

(a) Use Jordan canonical form to prove that every square matrix over the complex field is similar to its transpose.

(b) Use rational canonical form to find all possible similarity types of 4×4 matrix A over $\mathbb{Q}$, the rational field, which satisfy the equation $A^2 + A^4 = 0$.