DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
16 MAY 2008

Problem 1 (20 points) Determine necessary and sufficient conditions that a finitely generated abelian group A be semi-simple.

Problem 2 (20 points) Suppose G is a group. Show that there is a unique group homomorphism $\alpha : G \to A$ where A is abelian such that for any abelian group B, there is an isomorphism

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(G, B) \tag{1}$$

induced by α.

Problem 3 (20 points) Let m be an integer, $m \neq 0$. Consider the exact sequence

$$\textbf{(E)} \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0. \tag{2}$$

Let A be an abelian group.

- Determine T in the sequence

$$0 \longrightarrow T \longrightarrow \mathbb{Z} \otimes A \xrightarrow{m} \mathbb{Z} \otimes A \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes A \longrightarrow 0. \tag{3}$$

  so that the sequence is exact.
- Determine H in the sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, A) \longrightarrow \mathrm{Hom}(\mathbb{Z}, A) \xrightarrow{m} \mathrm{Hom}(\mathbb{Z}, A) \longrightarrow H \longrightarrow 0. \tag{4}$$

  so that the sequence is exact.

Problem 4 (20 points) Let $\Bbbk$ be a field. Let q be an indeterminate over this field. The ring $\Bbbk[q]$ is the ring of polynomials in the indeterminate q and $\Bbbk(q)$ is the field of fractions of this polynomial ring.

- Show that $\mathrm{Hom}_{\Bbbk[q]}(\Bbbk(q), \Bbbk(q)) \equiv \Bbbk(q)$.
- Show that $\Bbbk(q)$ is a flat but not projective $\Bbbk[q]$-module.
- Show that $\Bbbk(q)/\Bbbk[q]$ is an injective $\Bbbk[q]$-module.

Problem 5 (20 points) Suppose that p is a prime integer. Let $\mathbb{F}_p$ be the field with p elements. Let K be a finite field extension of $\mathbb{F}_p$ of degree n. Let $F : K \to K$ be the Frobenius map defined by $F(x) = x^p$.

- Show that F is a linear tranformation of the $\mathbb{F}_p$ vector space K.
- Show that $F^n = I$, the identity map on K.
- In case p does not divide n, find the Jordan Normal Form for F.