DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
MAY 2007

Problem 1 Let R be a ring.

- **a.** (10 points) Define when an R-module M is *Artinian* and when it is *Noetherian.*
- **b.** (10 points) Let $R = \mathbb{Z}$, the ring of integers. Show that every Noetherian $\mathbb{Z}$-module is Artinian, or give a counter example.
- **c.** (10 points) Let M be an R-module and $N \subset M$ be a submodule. Show that if both N and M/N are Artinian then M is Artinian.

Problem 2

- **a.** (10 points) Let G be an Abelian group of order mn, where m and n are arbitrary positive integers. Show that there is a subgroup and a quotient group of G of order m.
- **b.** (10 points) Let $R = \mathbb{Z}/45\mathbb{Z}$. Find all finitely-generated R-modules (list without repetitions).

Problem 3 Let M and N be $\mathbb{Z}$-modules. In this problem all $\otimes$'s and Hom's are over $\mathbb{Z}$.

- **a.** (10 points) Does M being projective imply $M \otimes N$ being projective? Justify.
- **b.** (10 points) Does M being injective imply $M \otimes N$ being injective? Justify.
- **c.** (10 points) Simplify $\mathrm{Hom}(((\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Q}) \otimes ((\mathbb{Z}/9\mathbb{Z}) \oplus 2\mathbb{Z}), \mathbb{R} \oplus (\mathbb{Z}/9\mathbb{Z}))$.

Problem 4 Let $\mathbb{C}[t]$ be the ring of polynomials in t with complex coefficients. A $\mathbb{C}[t]$-module can be described by giving a vector space V (a $\mathbb{C}$-module) together with a linear operator in V (the t-action).

- **a.** (10 points) Find all simple $\mathbb{C}[t]$-modules.
- **b.** (10 points) Find a Jordan-Hölder filtration of the $\mathbb{C}[t]$-module $\left(\mathbb{C}^3, t \mapsto \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$.