DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
MAY 23, 2006

Problem 1

a. (2 points) Let R be a ring. Define a *short exact sequence* of R-modules, and say what it means for the short exact sequence to *split*.

b. (4 points) In case $R = \mathbb{F}$ is a field, give an example of a short exact sequence of R-modules that is not split or show that such an example does not exists.

c. (4 points) In case $R = \mathbb{Z}$ is ring of the integers, give an example of a short exact sequence of R-modules that is not split or show that such an example does not exists.

Problem 2

a. (2 points) Let R be a ring. Define a *simple* and a *semi-simple* R-module.

b. (3 points) Let $\Bbbk$ be a field, and give $M = \Bbbk^2$ the structure of a $\Bbbk[q]$-module (q an indeterminate) by fixing a 2×2 matrix A over $\Bbbk$. Give examples of A for which M is semi-simple and for which it is *not* semi-simple.

c. (15 points) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a semisimple $\mathbb{C}$-algebra.

d. (15 points) Suppose that $K = \mathbb{F}_2(q)$ is the field of rational functions in the variable q over the field of two elements $\mathbb{F}_2$. Let L be the extension of degree 2 over K given by $L = K[X]/(X^2 - q)$. Show that $L \otimes_K L$ is NOT a semi-simple algebra.

Problem 3 (20 points)

Suppose that $\alpha, \beta : \mathbb{Z}^3 \to \mathbb{Z}^3$ are homomorphisms whose matrixes with respect to the standard bases of $\mathbb{Z}^3$ are given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & -8 \\ -1 & 2 & -1 \\ 0 & -2 & 4 \end{pmatrix},$$

respectively. Determine the four abelian groups $\operatorname{Ker}\alpha$, $\operatorname{Coker}\alpha$, $\operatorname{Ker}\beta$, $\operatorname{Coker}\beta$.

Problem 4

a. (15 points) Let $S \to R$ be a homomorphism of rings and consider R as an R-S-bimodule using the ring homomorphism. If $A = A_R$ is a right R-module and $C = {}_SC$ a left S-module, show that

$$\operatorname{Hom}_S(A \otimes_R R, C) \simeq \operatorname{Hom}_R(A, \operatorname{Hom}_S(R, C)).$$

b. (5 points) Formulate the definition of injectivity of an R-module Q in terms of the functor $M \mapsto \operatorname{Hom}_R(M, Q)$.

c. (15 points) Suppose $\Bbbk$ is a field, and let R be a $\Bbbk$-algebra (there is a ring homomorphism $k \to R$, mapping to the center of R). Let V be a k-vector space. Prove that $\operatorname{Hom}_{\Bbbk}(R, V)$ is an injective left and right R-module.