DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
JANUARY 2012

Problem 1 (20 points) Let R be a ring (with identity, as always).

(a) Explain what is meant by saying that a short exact sequence *splits*. Given an example of a ring R and a non split short exact sequence of R modules.

(b) Let $\Bbbk$ be a field. Show that every short exact sequence of vector spaces is split.

(c) With $\Bbbk$ still a field, let p, q, r be non-negative integers and suppose we have a short exact sequence

$$0 \to \Bbbk^p \to \Bbbk^q \to \Bbbk^r \to 0.$$

What can you say about the relation between the integers p, q, r in this case? Explain.

Problem 2 (20 points) Let R be a ring, with identity.

(a) Let us define a *projective* R-module P as one that is a direct summand of a free module: $P \oplus M = F$ for some module M and a free module F. Show that any short exact sequence

$$0 \to A \to B \to P \to 0$$

splits if P is projective.

(b) Let $0 \to A \xrightarrow{f} B \xrightarrow{\pi} C \to 0$ be a short exact sequence of R-modules. Suppose that we are given homomorphisms $\alpha_1 \colon P_1 \to A$, $\alpha_2 \colon P_2 \to C$, where P_i are projective. Show that there is a projective module P and a homomorphism $\alpha \colon P \to B$ such that the following diagram commutes for some homomorphisms d_1, d

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{d_1} & P & \xrightarrow{d} & P_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{\pi} & C & \longrightarrow & 0
\end{array}$$

and such that the top row is also exact. Explain in detail what P is (and why is it projective!), and what the maps d_1, d, α are.

(c) Let $M = \mathbb{Z}/3\mathbb{Z}$ and consider M as a module over the rings R given below.
- If $R = \mathbb{Z}$ is M free? Projective?
- If $R = \mathbb{Z}/3\mathbb{Z}$ is M free? Projective?
- If $R = \mathbb{Z}/6\mathbb{Z}$ is M free? Projective?

In each case explain how M is a module over the given ring R.

Problem 3 (20 points) Let R, S, T be not necessarily commutative rings with identity.

(a) If ${}_RM_S$ and ${}_SN_T$ are bimodules as shown, describe in detail the module structure of $M \otimes_S N$ and justify your answer.

(b) Given a bimodule ${}_RM_S$, prove that there is an isomorphism of left R-modules $M \otimes_S S \simeq M$.

(c) Given bimodules L_R, ${}_RM_S$, N_S, show that there is an isomorphism of abelian groups

$$\mathrm{Hom}_S(L \otimes_R M, N) \simeq \mathrm{Hom}_R(L, \mathrm{Hom}_S(M, N)).$$

Problem 4 (20 points) Let R be a ring with identity and let M be an R-module such that $M = M_1 + M_2 + \cdots + M_k$ where the M_i are simple submodules of M.

(a) Prove that M is the direct sum of certain of the M_i.

(b) If N is a submodule of M, prove that N is a direct summand of M, i.e., $M = N \oplus L$ for some submodule L.

(c) Prove that every submodule and quotient module of M is a direct sum of simple submodules isomorphic with certain M_i's.

Problem 5 (20 points) Let G be a finite group and $\mathbb{C}$ the complex field.

(a) Explain in detail how the structure of the group algebra $\mathbb{C}G$ determines the irreducible $\mathbb{C}$-representations of G.

(b) Prove that $|G| = n_1^2 + n_2^2 + \cdots + n_k^2$ where the n_i are the degrees of the irreducible $\mathbb{C}$-representations of G.

(c) Let G be the symmetric group of degree 3. Find the degrees of the irreducible $\mathbb{C}$-representations of G and describe the corresponding representations.