DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
JANAURY 2008

Problem 1 (20 points) Let R be a ring and let A be a left R–module.

A) Prove that $Hom_R(A, A)$ is a ring via composition.

B) Prove that the ring $Hom_R(A^{\oplus 2}, A^{\oplus 2})$ is isomorphic (as a ring) to $M_{2\times 2}(Hom_R(A, A))$.

Problem 2 (20 points) Let R be a ring. Let A be a right R–module and let

$$0 \to M'' \to M \to M' \to 0$$

be a short exact sequence of left R–modules. The following questions are related to the sequence

$$(I) \qquad 0 \to A \otimes_R M'' \to A \otimes_R M \to A \otimes_R M' \to 0$$

A) Show by explicit example that I need not be a short exact sequence.

B) Prove that if A is a projective right R–module than I is exact.

C) Prove that if M' is a projective left R–module then I is exact.

D) Prove that if M'' is an injective left R–module then I is exact.

Problem 3 (20 points) Let $\Bbbk$ be a field of characteristic $p > 0$. Let G be a finite group.

A) Demonstrate that if the order of G is relatively prime to p, then the group ring $\Bbbk G$ is semi-simple.

B) Show by explicit example, with details included, that this is not necessarily the case if p divides the order of G.

Problem 4 (20 points) Suppose A is a commutative ring with 1. Suppose T is a non–empty subset of A that satisfies the properties:

i) If $s, t \in T$, then $st \in T$. ii) $0 \notin T$.

A) Prove that an ideal M of A maximal with respect to the property that $M \cap T = \emptyset$ is a prime ideal.

For such a T, define the set

$$T' = \{s \mid \text{there exists a } t \text{ such that } st \in M\}.$$

B) Show that

(1) $T \subseteq T'$ and that T' satisfies the same property as T.

(2) $(T')' = T'$.

(3) The induced ring homomorphism $T^{-1}A \to T'^{-1}A$ is an isomorphism.

Problem 5 (20 points) Let $\Bbbk$ be a field.

A) Show that a finite subgroup G of the multiplicative group $\Bbbk^\times$ of units of $\Bbbk$ is cyclic.

B) Suppose $[G : 1] > 1$. Show that

$$(1) \qquad \sum_{g \in G} g = 0.$$