DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE
EXAMINATION
JANUARY 2007

Problem 1 Let R be a ring.

a. (10 points) Define when an R-module is *projective*. In case $R = \mathbb{Z}$, the ring of integers, find all finitely generated projective $\mathbb{Z}$-modules.

b. (10 points) Define when an R-module is *injective*. In case $R = \mathbb{Z}$, the ring of integers, show an example of an injective $\mathbb{Z}$-module. Justify your answer.

Problem 2 (25 points) Let m, n be integers ≥ 1. Find the tensor products

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} m\mathbb{Z}, \quad \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}.$$

Hint: $0 \to m\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$ is exact.

Problem 3 Let R be a ring.

a. (10 points) Define when an R-module is *simple*. In case $R = \mathbb{Z}$, the ring of integers, find all simple $\mathbb{Z}$-modules.

b. (10 points) Let $R = \mathbb{C}[[x]]$. Show that R has a unique maximal ideal, or, equivalently, that all non-units in R form an ideal.

c. (10 points) Let $R = \mathbb{C}[[x]]$. Find all simple R modules.

Problem 4 (25 points) Let $\mathbb{F}_3$ be the field with 3 elements. Describe all conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_3)$.