DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
JANUARY 21ST, 2006

Problem 1. (25 points) Suppose A is a commutative ring with 1. Suppose T is a subset of A that satisfies the properties: $i)$ $0 \notin T$, $ii)$ $1 \in T$, and $iii)$ If $s, t \in T$, then their product $st \in T$.
Prove that an ideal M of A maximal with respect to the property that $M \cap T = \emptyset$ is a prime ideal.

Problem 2. (25 points) Let $\mathbb{k}$ be a field. Suppose G, H are groups and let $\mathbb{k}[G], \mathbb{k}[H]$ be the group algebras over $\mathbb{k}$. Prove that there is an isomorphism

$$\mathbb{k}[G \times H] \cong \mathbb{k}[G] \otimes_{\mathbb{k}} \mathbb{k}[H].$$

Problem 3. Let A be a finite Abelian group.

a. (10 points) Show that the rational group algebra $\mathbb{Q}[A]$ is a product of fields.

b. (15 points) In case $A = \mathbb{Z}/2\mathbb{Z}$ and $A = \mathbb{Z}/3\mathbb{Z}$ identify the fields that occur in $\mathbb{Q}[A]$.

Problem 4. Let A now be a commutative ring with 1 and M an A-module.

a. (5 points) Define what it means to say that M is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

b. (10 points) In case $A = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$ explain whether or not M is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

c. (10 points) In case $A = \mathbb{Z}$ and $M = \mathbb{Q}$ explain whether or not M is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.