DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
AUGUST 2009

Problem 1 Let R be a commutative ring with 1 and M an R-module. Recall that M is defined to be R-flat if for every ideal $I \subset R$ the sequence

$$0 \to I \otimes_R M \to R \otimes_R M$$

is left exact, i.e., the map $I \otimes_R M \to R \otimes_r M$ is injective.

- **a.** (7 points) Prove that $\mathbb{Q}$, the field of rational numbers, is $\mathbb{Z}$-flat, where $\mathbb{Z}$ is the ring of integers.
- **b.** (6 points) Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$ module.
- **c.** (7 points) Is the quotient $\mathbb{Q}/\mathbb{Z}$ $\mathbb{Z}$-flat? Justify your answer.

Problem 2 Let R be a commutative ring with 1.

- **a.** (7 points) Define what an injective $\mathbb{Z}$-module is, and show that both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.
- **b.** (7 points) For any R-module M define the dual M^* to be $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. The dual of a module is itself an R-module. Prove that for any module M the map

$$\phi_M : M \to (M^*)^*,$$

defined by $\phi_M(x)(f) = f(x)$, for $x \in M, f \in M^*$, is injective.
- **c.** (6 points) if P is a projective module its dual P^* is injective. Use this to prove that any R-module can be embedded into an injective R-module.

Problem 3

- **a.** (8 points) Prove that an Artinian integral domain is a field.
- **b.** (4 points) Show that any prime ideal in an Artinian ring is maximal.
- **c.** (8 points) Let $R = \frac{\mathbb{Q}[x,y,x]}{(x^3,y^4,z^5)}$. Let M be any finitely generated R-module. Prove that M has finite length over R.

Problem 4 For parts **a.**, **b.** $\Bbbk'$ is a field extension of a field $\Bbbk$.

a. (7 points) Let A be a $n \times n$ matrix over $\Bbbk$. Prove that the invariants of A are the same as over $\Bbbk'$.

b. (5 points) Let now A, B be two $n \times n$ matrices over $\Bbbk$ and assume there is an invertible matrix C' over the extension $\Bbbk'$ such that $B = C'A(C')^{-1}$. Show that there is an invertible matrix C over $\Bbbk$ such that $B = CAC^{-1}$. (Hint: you can use part **a.**)

c. (8 points) Let $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Find the minimal polynomial of A over $\mathbb{Q}$. Show that A is similar to $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

Problem 5 Let R be a commutative ring and M an R-module.

a. (4 points) Define what it means to say that M is a *cyclic* module, and what it means to say that M is *simple* (=*irreducible*).

b. (4 points) Show that all simple modules are cyclic.

c. (5 points) It is not true that all cyclic modules are simple. Give an example of this situation.

d. (7 points) Let C_n be the cyclic group of order $n > 0$ and let $R = \mathbb{C}[C_n]$ be the complex group algebra of C_n. Find all cyclic R-modules.