Department of Mathematics
501 Comprehensive Exam
August 30, 2008

1. Let A be a commutative ring with 1 and let M be an A-module.

a. (*5 points*) Define what it means to say that M is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

b. (*10 points*) When $A = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$ explain whether of not M is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

c. (*10 points*) When $A = \mathbb{Z}$ and $M = \mathbb{Q}$ explain whether or not M is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

2 . Let A be a commutative ring with 1 and let M be an A-module..

a. (*5 points*) Define what it means for the module M to be a Noetherian module, an Artinian module, a finitely generated module.

b. (*10 points*) Let M be a Noetherian A-module and $\alpha : M \to M$ a module homomorphism. Prove that if u is surjective then u is an isomorphism.

c. (*10 points*) Prove that if M and N are finitely generated A modules then $M \otimes_A N$ is again a finitely generated A-module.

3. Let G be a finite abelian group.

a. (*15 points*) Show that the rational group algebra $\mathbb{Q}[G]$ is a product of fields.

b. (*10 points*) In the cases $G = \mathbb{Z}/2\mathbb{Z}$ and $G = \mathbb{Z}/3\mathbb{Z}$ identify the fields that occur in $\mathbb{Q}[G]$.

4. Let $\alpha : V \to V$ be a linear transformation of a finite vector space over a field F.

a. (*10 points*) Prove that α is diagonalizable (there exists a basis of V such that the associated matrix of α is a diagonal matrix) if and only if the minimal polynomial of α splits into linear factors (in F) with no repeated roots.

b. (*15 points*) Prove that if α is diagonalizable then V is a semi-simple $F[T]$ module (where the action is induced via α in the customary manner, given $f \in F[T]$ and $v \in V$, $f * v = f(\alpha)(v)$).