DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE
EXAMINATION
AUGUST 2007

Problem 1 Let R be a ring.

- **a.** (10 points) Define when an R-module M is *simple*. When is R itself simple?
- **b.** (10 points) Suppose R is a matrix ring over a division ring Δ: $R = \mathrm{Mat}_{n\times n}(\Delta)$. Show that R is simple. Calculate the center $Z(R)$ of R.
- **c.** (10 points) Still let $R = \mathrm{Mat}_{n\times n}(\Delta)$. Describe all simple R-modules. Justify.

Problem 2 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules over some ring. True or false (if true – prove, if false – provide a counterexample):

- **a.** (10 points) A and C are Noetherian $\Rightarrow$ B is Noetherian;
- **b.** (10 points) A and C are projective $\Rightarrow$ B is projective;
- **c.** (10 points) A and C are semisimple $\Rightarrow$ B is semisimple.

Problem 3 (10 points) Write $\mathbb{Z}^3/L$, where $L = \mathbb{Z}(3,2,1)+\mathbb{Z}(2,-2,2)$, as a direct sum of cyclic abelian groups.

Problem 4 Let $\mathbb{R}, \mathbb{C}$ be the fields of real and complex numbers. Simplify:

- **a.** (10 points) $\mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[x]$
- **b.** (10 points) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]$
- **c.** (10 points) $\big(\mathbb{R}[x]/(x^2-1)\big) \otimes_{\mathbb{R}[x]} \big(\mathbb{R}[x]/(x^3-1)\big)$