DEPARTMENT OF MATHEMATICS
MATHEMATICS 501 COMPREHENSIVE EXAMINATION
AUGUST, 2006

Problem 1 Let R be a ring and V an R-module that is a direct sum: $V = V_1 \oplus V_2$, for R-modules V_1 and V_2. Let $U \subset V$ be a submodule, and $U_i = U \cap V_i$, $i = 1, 2$.

a.(10 points): Show that in general we need not have $U = U_1 \oplus U_2$.

b.(10 points): In case $R = \Bbbk[h]$, for $\Bbbk$ a field of characteristic zero, and h a variable, let $V_i = \{v \in V \mid hv = \lambda_i v\}$, with $\lambda_1 \neq \lambda_2 \in \Bbbk$. Show that in this case we *do* have $U = U_1 \oplus U_2$.

Problem 2 Let R be a commutative ring with identity.

a.(10 points): Prove from the definitions that a direct sum of projective R-modules is projective.

b.(10 points): Suppose R is an integral domain. Show that every R-module is projective if and only if R is a field.

Problem 3

a.(10 points): Let $\Bbbk$ be a field, and let $M = \Bbbk^2$ be the space of column vectors of size 2. Give M the structure of a $\Bbbk[h]$-module (h an indeterminate) by fixing a 2×2 matrix A over $\Bbbk$. (So $f(h)$ acts on M by $f(h).m = f(A)m$). In case $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is M cyclic? Simple? Semisimple? Justify your answers.

b.(10 points): Let R be a commutative ring with 1 and $n \neq 0 \in R$ a nilpotent element, i.e., for some $k > 1$ we have $n^k = 0$. Can R be semisimple over itself?

Problem 4 (20 points) Demonstrate from first principles that a submodule of a finitely generated free module over a principal ideal domain is a free module.

Problem 5 Let h be an indeterminate over the ring $\mathbb{Q}$. Determine the following $\mathbb{Q}[h]$ modules:

(1) **10 points** $\left(\mathbb{Q}[h]/(h^3 - h^2 - h + 1)\right) \otimes_{\mathbb{Q}[h]} \left(\mathbb{Q}[h]/(h^3 + h^2 - h - 1)\right)$

(2) **10 points** $\left(\mathbb{Q}[h]/(h^2 + 1)\right) \otimes_{\mathbb{Q}[h]} \left(\mathbb{Q}[h]/(h^2 + h + 1)\right)$