MATH 500 — May 2018

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. Let P be a Sylow p-subgroup of a finite group G and let N be a normal subgroup of G, such that $P \cap N \neq \{e\}$. Prove that $P \cap N$ is a Sylow p-subgroup of N.

2. Let S_5 be the symmetric group in 5 elements and let $\phi : S_5 \to G$ be a group homomorphism. Classify the image $\phi(S_5)$, i.e., list all the possibilities for $\phi(S_5)$ up to isomorphism.

3. Let $T : \mathbb{Q}^4 \to \mathbb{Q}^4$ be the $\mathbb{Q}$-linear transformation which relative to some basis is represented by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

Find the rational canonical form for T.

4. Let $\langle(11, 13)\rangle$ be the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by the element $(11, 13)$. Show that the quotient group

$$(\mathbb{Z} \oplus \mathbb{Z})/\langle(11, 13)\rangle$$

is torsion free.

5. (a) Find the Galois group of the polynomial $p(x) = x^3 - 10$ over the field $K = \mathbb{Q}(\sqrt{2})$.

   (b) Let $q(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of prime degree $p \geq 2$. Show that if $q(x)$ has exactly two non-real roots (i.e., two complex roots) then the Galois group of $q(x)$ is isomorphic to S_p.