MATH 500 — MAY 2017

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. (a) Let G be a group of order $2n$ $(n > 0)$. Show that G has at least one element of order 2.

   (b) Consider the action of $\mathbb{Z}_2 \cong (\{1, -1\}, \times)$ on S_4 by automorphisms, where -1 acts by conjugation by the transposition $(1\ 2)$. Is the semi-direct product $\mathbb{Z}_2 \ltimes S_4$ a nilpotent group? Solvable?

2. (a) Give an example of a finite field of order 27.

   (b) Suppose a commutative ring R with identity has order 27. List all possible values of the characteristic of R, and give examples to show that all the values you list are attained.

3. Let $\mathbb{R}^\infty = \oplus_{k=1}^{+\infty}\mathbb{R}$ (direct sum of $\mathbb{R}$-modules) and let $R = \mathrm{End}(\mathbb{R}^\infty)$ be the ring of all $\mathbb{R}$-linear transformations from $\mathbb{R}^\infty$ to itself. Show that R is isomorphic to $R \oplus R$ as a left R-module (so R, viewed as a left R-module, has a basis with 2 elements!).

4. Given an example of an integral domain D, where every irreducible element is prime, and which admits an infinite chain of ascending principal ideals:

$$\langle d_1 \rangle \subsetneq \langle d_2 \rangle \subsetneq \langle d_3 \rangle \subsetneq \cdots \subsetneq \langle d_n \rangle \subsetneq \cdots$$

   What can you say about prime factorizations in this domain?

5. Consider the extension $L = \mathbb{Q}(\sqrt[3]{5}, i)$ of $\mathbb{Q}$. Find all subfields $\mathbb{Q} \subsetneq M \subsetneq L$ which are normal extensions of $\mathbb{Q}$.