MATH 500 — MAY 2016

Five problems, 20 points each. Maximum 100 points.

1. Classify (up to isomorphism) the finite groups G that are both solvable and simple.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which is represented relative to the canonical basis by the matrix:
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$
Consider on $\mathbb{R}^3$ the $\mathbb{R}[x]$-module structure defined by T. Show that there exists $p_0(x) \in \mathbb{R}[x]$ and an isomorphism of $\mathbb{R}[x]$-modules:
$$\mathbb{R}^3 \simeq \mathbb{R}[x]/\langle p_0(x) \rangle.$$

3. Recall that a real number is *algebraic*, if it is the root of some polynomial $p(x) \in \mathbb{Q}[x]$. Show that the set $A \subset \mathbb{R}$ of real algebraic numbers is an algebraic extension of $\mathbb{Q}$. What is $\dim_{\mathbb{Q}} A$?

4. Give examples of:

   (a) a commutative ring with 1 that is not an integral domain,

   (b) an integral domain that is not a UFD,

   (c) a UFD that is not a PID.

   (d) A prime ideal that is not maximal.

   (e) An ideal that is not prime.

   Justify, for each of your examples, that it has the properties claimed.

5. Compute, with proof, the Galois group of $g(x) = x^3 - 4x + 1 \in \mathbb{Q}[x]$.