# MATH 500 COMP EXAM – MAY 2019

Five problems, 20 points each. Maximum 100 points.

(1) (a) Let $p$ be the smallest prime dividing the order of a finite group $G$. If $H$ is a subgroup of $G$ of index $p$, prove that $H$ is normal in $G$.
(b) Show that any group of order 77 is cyclic.

(2) Let $q$ be a prime power and let $\mathbb{F}_q$ be a finite field with $q$ elements. Let $\mathrm{GL}_2(\mathbb{F}_p)$ be the (finite) group of invertible $2 \times 2$ matrices with coefficients in $\mathbb{F}_p$.
(a) Show that there is a group homomorphism $\mathrm{GL}_2(\mathbb{F}_q) \to S_{q+1}$ with kernel equal to the subgroup of scalar matrices $Z = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{F}_q,\ a \neq 0 \right\}$. (Hint: Construct an action of $\mathrm{GL}_2(\mathbb{F}_q)$ on the set of one-dimensional subspaces of $\mathbb{F}_q^2$.)
(b) Use part (a) to prove that $\mathrm{GL}_2(\mathbb{F}_3)$ is solvable and that $\mathrm{GL}_2(\mathbb{F}_4)$ is not solvable. You may use without proof that $\mathrm{GL}_2(\mathbb{F}_q)$ has cardinality $(q^2-1)(q^2-q)$.

(3) Let $M$ be the quotient abelian group $\mathbb{Z}^4/A$, where $A$ is the subgroup of $\mathbb{Z}^4$ generated by the elements $(1,1,1,1), (0,1,1,0)$, and $(1,2,-1,0)$.
(a) Determine the structure of $M$.
(b) How many non-trivial homomorphisms are there $M \to \mathbb{Z}/5$?

(4) Let $k$ be a field, and consider the element $D = \det \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ in the polynomial ring $k[x,y,z,w]$.
(a) Show that $D$ is irreducible.
(b) Show that $k[x,y,z,w]/D$ is not a UFD.

(5) Let $K$ be the splitting field of $x^6 + 3$ over $\mathbb{Q}$.
(a) Compute the Galois group of $K$ over $\mathbb{Q}$.
(b) How many subfields of $K$ are there, which have degree 3 over $\mathbb{Q}$?